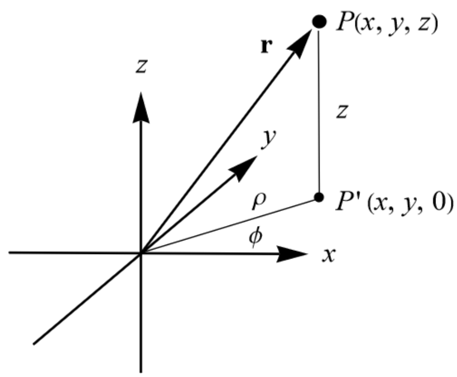
Problem 1.47

Let the position of a point P in three dimensions be given by the vector $\mathbf{r} = (x, y, z)$ in rectangular (or Cartesian) coordinates. The same position can be specified by **cylindrical polar coordinates**, ρ, ϕ, z, which are defined as follows: Let P' denote the projection of P onto the xy plane; that is, P' has Cartesian coordinates $(x, y, 0)$. Then ρ and ϕ are defined as the two-dimensional polar coordinates of P' in the xy plane, while z is the third Cartesian coordinate, unchanged. **(a)** Make a sketch to illustrate the three cylindrical coordinates. Give expressions for ρ, ϕ, z in terms of the Cartesian coordinates x, y, z. Explain in words what ρ is ("ρ is the distance of P from ______"). There are many variants in notation. For instance, some people use r instead of ρ. Explain why this use of r is unfortunate. **(b)** Describe the three unit vectors $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$ and write the expansion of the position vector $\mathbf{r}$ in terms of these unit vectors. **(c)** Differentiate your last answer twice to find the cylindrical components of the acceleration $\mathbf{a} = \ddot{\mathbf{r}}$ of the particle. To do this, you will need to know the time derivatives of $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. You could get these from the corresponding two-dimensional results (1.42) and (1.46), or you could derive them directly as in Problem 1.48.

Solution

Part (a)

Below is a sketch that illustrates the three cylindrical coordinates (ρ, ϕ, z).



The relationships between the cylindrical and rectangular coordinates are derived in Problem 1.42.

$$x^2 + y^2 = \rho^2 \quad \rightarrow \quad \rho = \sqrt{x^2 + y^2}$$

$$\tan\phi = \frac{y}{x} \quad \Rightarrow \quad \phi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \text{ and } y \text{ are positive (Quadrant I)} \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \text{ is negative and } y \text{ is positive (Quadrant II)} \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \text{ and } y \text{ are negative (Quadrant III)} \\ \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \text{ is positive and } y \text{ is negative (Quadrant IV)} \end{cases}.$$

$$z = z \quad \rightarrow \quad z = z$$

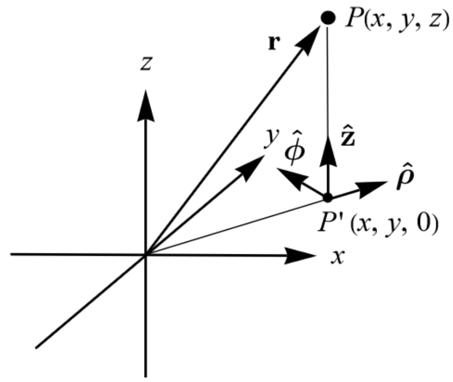
ρ is the distance of (x, y, z) from the z-axis—the perpendicular distance, that is. ϕ is the angle measured counterclockwise from the x-axis in the xy-plane. z is the vertical height from the xy-plane. The problem with using r for ρ (as in Problem 1.42) is that r is commonly used in physics texts to represent the distance from the origin to (x, y, z).

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

In calculus texts it's the other way around: $\rho = \sqrt{x^2 + y^2 + z^2}$ and $r = \sqrt{x^2 + y^2}$. One can tell from context what meaning r has.

Part (b)

The unit vectors in cylindrical coordinates are illustrated below.



$\hat{\boldsymbol{\rho}}$ points radially outward from the z-axis; $\hat{\boldsymbol{\phi}}$ is perpendicular to both $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$, pointing in the direction of increasing ϕ; and $\hat{\mathbf{z}}$ points in the direction of the z-axis.

$$\begin{aligned}
\hat{\boldsymbol{\rho}} &= \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}|} \\
&= \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + 0^2}} \\
&= \frac{x}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2 + y^2}}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= \frac{x}{\rho}\,\hat{\mathbf{x}} + \frac{y}{\rho}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}} \\
&= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= \hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} \\
&= \hat{\mathbf{z}} \times (\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{x}}) + \sin\phi\,(\hat{\mathbf{z}} \times \hat{\mathbf{y}}) + 0\,(\hat{\mathbf{z}} \times \hat{\mathbf{z}}) \\
&= \cos\phi\,(\hat{\mathbf{y}}) + \sin\phi\,(-\hat{\mathbf{x}}) + 0\,(\mathbf{0}) \\
&= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}
\end{aligned}$$

$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

In terms of these cylindrical unit vectors,

$$\begin{aligned}
\mathbf{r} &= x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} \\
&= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}\,\hat{\mathbf{x}} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}\right) + z\,\hat{\mathbf{z}} \\
&= \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}.
\end{aligned}$$

Part (c)

The aim here is to differentiate $\mathbf{r}$ with respect to t twice in order to obtain $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$. Find the first derivative.

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{d\mathbf{r}}{dt} \\
&= \frac{d}{dt}(\rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}) \\
&= \frac{d}{dt}(\rho\,\hat{\boldsymbol{\rho}}) + \frac{d}{dt}(z\,\hat{\mathbf{z}}) \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d\hat{\boldsymbol{\rho}}}{dt} + \frac{dz}{dt}\,\hat{\mathbf{z}} + z\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) + \frac{dz}{dt}\,\hat{\mathbf{z}} + z\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\left[\frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\sin\phi\,\hat{\mathbf{y}})\right] + \frac{dz}{dt}\,\hat{\mathbf{z}} + z\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\left[\frac{d}{dt}(\cos\phi)\,\hat{\mathbf{x}} + \cos\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\sin\phi)\,\hat{\mathbf{y}} + \sin\phi\,\frac{d\hat{\mathbf{y}}}{dt}\right] + \frac{dz}{dt}\,\hat{\mathbf{z}} + z\,\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\left[\left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \cos\phi\,\underbrace{\frac{d\hat{\mathbf{x}}}{dt}}_{=\mathbf{0}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} + \sin\phi\,\underbrace{\frac{d\hat{\mathbf{y}}}{dt}}_{=\mathbf{0}}\right] + \frac{dz}{dt}\,\hat{\mathbf{z}} + z\,\underbrace{\frac{d\hat{\mathbf{z}}}{dt}}_{=\mathbf{0}}
\end{aligned}$$

The derivative of any Cartesian unit vector with respect to time is zero.

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\left[\left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}}\right] + \frac{dz}{dt}\,\hat{\mathbf{z}} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d\phi}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) + \frac{dz}{dt}\,\hat{\mathbf{z}} \\
&= \frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \frac{dz}{dt}\,\hat{\mathbf{z}}
\end{aligned}$$

Find the second derivative.

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d\dot{\mathbf{r}}}{dt} \\
&= \frac{d}{dt}\left(\frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}} + \rho\,\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \frac{dz}{dt}\,\hat{\mathbf{z}}\right) \\
&= \frac{d}{dt}\left(\frac{d\rho}{dt}\,\hat{\boldsymbol{\rho}}\right) + \frac{d}{dt}\left(\rho\,\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}}\right) + \frac{d}{dt}\left(\frac{dz}{dt}\,\hat{\mathbf{z}}\right) \\
&= \frac{d}{dt}\left(\frac{d\rho}{dt}\right)\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d}{dt}\left(\frac{d\phi}{dt}\right)\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \frac{d}{dt}\left(\frac{dz}{dt}\right)\hat{\mathbf{z}} + \frac{dz}{dt}\frac{d\hat{\mathbf{z}}}{dt} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\frac{d\hat{\boldsymbol{\phi}}}{dt} + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} + \frac{dz}{dt}\underbrace{\frac{d\hat{\mathbf{z}}}{dt}}_{=\mathbf{0}} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\,\frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}) + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\left[\frac{d}{dt}(\cos\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\sin\phi\,\hat{\mathbf{y}})\right] + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\left[\frac{d}{dt}(-\sin\phi\,\hat{\mathbf{x}}) + \frac{d}{dt}(\cos\phi\,\hat{\mathbf{y}})\right] + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\left[\frac{d}{dt}(\cos\phi)\,\hat{\mathbf{x}} + \cos\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\sin\phi)\,\hat{\mathbf{y}} + \sin\phi\,\frac{d\hat{\mathbf{y}}}{dt}\right] + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} \\
&\qquad + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\left[\frac{d}{dt}(-\sin\phi)\,\hat{\mathbf{x}} - \sin\phi\,\frac{d\hat{\mathbf{x}}}{dt} + \frac{d}{dt}(\cos\phi)\,\hat{\mathbf{y}} + \cos\phi\,\frac{d\hat{\mathbf{y}}}{dt}\right] + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\left[\left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \cos\phi\,\overbrace{\frac{d\hat{\mathbf{x}}}{dt}}^{=\mathbf{0}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} + \sin\phi\,\overbrace{\frac{d\hat{\mathbf{y}}}{dt}}^{=\mathbf{0}}\right] + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} \\
&\qquad + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\left[\left(-\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} - \sin\phi\,\underbrace{\frac{d\hat{\mathbf{x}}}{dt}}_{=\mathbf{0}} + \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}} + \cos\phi\,\underbrace{\frac{d\hat{\mathbf{y}}}{dt}}_{=\mathbf{0}}\right] + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}}
\end{aligned}$$

The derivative of any Cartesian unit vector with respect to time is zero.

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\left[\left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \left(\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}}\right] + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} \\
&\qquad + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d\phi}{dt}\left[\left(-\cos\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{x}} + \left(-\sin\phi\cdot\frac{d\phi}{dt}\right)\hat{\mathbf{y}}\right] + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} \\
&= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\frac{d\phi}{dt}(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} \\
&\qquad + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} - \rho\left(\frac{d\phi}{dt}\right)^2(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}}
\end{aligned}$$

The vectors in parentheses are unit vectors in cylindrical coordinates.

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d^2\rho}{dt^2}\,\hat{\boldsymbol{\rho}} + \frac{d\rho}{dt}\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} + \frac{d\rho}{dt}\frac{d\phi}{dt}\,\hat{\boldsymbol{\phi}} + \rho\,\frac{d^2\phi}{dt^2}\,\hat{\boldsymbol{\phi}} - \rho\left(\frac{d\phi}{dt}\right)^2\hat{\boldsymbol{\rho}} + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}} \\ &= \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right]\hat{\boldsymbol{\rho}} + \left(\rho\,\frac{d^2\phi}{dt^2} + 2\frac{d\rho}{dt}\frac{d\phi}{dt}\right)\hat{\boldsymbol{\phi}} + \frac{d^2z}{dt^2}\,\hat{\mathbf{z}}\end{aligned}$$

Therefore, since $\mathbf{a} = \ddot{\mathbf{r}}$, the components of acceleration in cylindrical coordinates are

$$\begin{cases} a_\rho = \dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\phi}{dt}\right)^2 \\ a_\phi = \rho\,\dfrac{d^2\phi}{dt^2} + 2\dfrac{d\rho}{dt}\dfrac{d\phi}{dt} \\ a_z = \dfrac{d^2z}{dt^2} \end{cases}.$$